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# A cosmotopological relation for a unified field theory

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I present an argument, based on the topology of the universe, why there are three generations of fermions. The argument implies a preferred unified gauge group of  $SU(5)$ , but with  $SO(10)$  representations of the fermions. The breaking pattern  $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$  is preferred over the pattern  $SU(5) \rightarrow SU(4) \times U(1)$ . On the basis of the argument one expects an asymmetry in the early universe microwave data, which might have been detected already.

## 1 Introduction

The standard model based on the gauge group  $SU(3) \times SU(2) \times U(1)$  with its complicated set of fermions, constrained by the anomaly is simply asking for a unification into a larger group. It is well known that such a unification is possible and quite natural. Given the fact that there is no convincing evidence for the existence of extra  $Z'$ s or  $W'$ s the known vector bosons point towards a unification within a group of rank 4, namely  $SU(5)$  [1].

However, at the latest with the discovery of neutrino masses, it has become clear that the natural unification for the fermions is within the group  $SO(10)$  [2], since each generation forms an irreducible spinor representation of  $SO(10)$ . So naively speaking the vector bosons and the fermions point toward a different form of unification. Of course the situation can be described through the breaking of the symmetry with a number of Higgs fields, but one would hope for a more fundamental explanation for this feature. Another fact of phenomenology is the existence of precisely three generations of fermions. It is natural to wonder whether the group question  $SU(5)$  versus  $SO(10)$  is related to the question of the number of generations. We are therefore looking for an argument to constrain the representation content and the gauge group of the theory. The only type of argument known that can give such constraints is based on some form of an anomaly. As anomalies are intimately related to topology, one is led to the question: what is the topology of space and time?

In typical Robertson-Walker metrics the topology of the world is a sphere or an open space. In higher dimensional cosmologies many shapes are possible. For instance one can take  $M_4 \times U^n(1)$ , a torus shape for the higher dimensions. One would like to describe the universe as starting in a higher dimensional space, where some of the dimensions dynamically shrink to become too small to be seen today. This idea is however difficult to realize in practice, as the Einstein equations tend to lead to an expansion of the universe in all directions. Therefore we start with the opposite assumption. We assume that the early universe was lower dimensional than it is now. To be more precise we assume a topology

$M_3 \times U(1)$  for space-time, where the radius of the third dimension is small in the early universe. Through the cosmological expansion the radius becomes very large, so that at the present epoch one finds a very large, essentially flat and isotropic universe. This behaviour can be realized in the Bianchi-I universes [3]. For a modern review on non-isotropic and/or non-homogeneous cosmological models, see [4].

Since the present universe is very large and flat, we have no direct information on its overall topology. However an indirect indication of the topology can be found through the appearance of matter fields, while not all matter fields can lead to a consistent quantum field theory for a given topology. For instance, in order to have spin-1/2 fields the Stiefel-Whitney class of the manifold has to vanish. Another example appears in three spacetime dimensions when a Chern-Simons mass term for gauge fields is present. For non-abelian gauge fields, invariance under large gauge transformations requires the mass to be quantized in units of the coupling constant[5-10]. When massive fermions are present, loop effects give rise to a finite renormalization of the Chern-Simons term[10], leading to restrictions on the number of fermion fields. For massless fermions the restrictions are unchanged, but then they come from a non-perturbative parity anomaly[11-14]. Closely related to the parity anomaly and of interest to our argument is the CPT anomaly[15], that can appear if the universe has a preferred direction, for instance when it has the topology  $M_3 \times U(1)$ . In this case a Chern-Simons like term can arise for the photon, due to the presence of Weyl-fermions in the theory.

$$L_{CS\,like} = m_{ph} n^\alpha \epsilon_{\alpha\beta\gamma\delta} A^\beta F^{\gamma\delta} \quad (1)$$

Here  $n^\alpha$  is the preferred direction in space and  $m_{ph}$  is the mass-like term for the photon. This term violates Lorentz invariance and CPT. It is called Chern-Simons like, because a true Chern-Simons term exists only in three dimensions. If this term is present, it gives rise to a number of interesting effects in the propagation of photons in spacetime. The CPT anomaly is actually a lifting to four dimensions of the three dimensional parity anomaly. The connection is most easily seen when one takes the radius of the  $U(1)$  to zero, thereby dimensionally reducing the theory. In this limit the four dimensional photon becomes a three dimensional Chern-Simons photon. The four dimensional Weyl-fermions become three dimensional Dirac fermions. The structure of the anomaly can therefore be analysed in the dimensionally reduced theory. For a didactic introduction to the issues involved we refer to [16].

## 2 The argument

Since we assume the early universe to be of the form  $M_3 \times U(1)$ , with a very small compactified  $U(1)$  in the early universe, we are interested in the consistency of the quantum field theory, reduced to three dimensions, ignoring the compactified dimension. Quantum anomalies in such space-times have been discussed before within the subject of Chern-Simons theories. In such theories Chern-Simons terms are a part of gravitational and Yang-Mills fields. However their coefficients are quantized. The matter fields in the theory give rise to loop-induced gravitational Chern-Simons terms, that in general will not satisfy the correct quantization rules. Thereby one can constrain the matter fields of the theory. In [17-20] corrections to the gravitational Chern-Simons term due to fermions were considered, in

[20, 21] the effects of the vector bosons were considered. The back-reaction of a gravitational Chern-Simons term on the vector bosons was considered in [22].

The gravitational action in three dimensions contains two terms. One is the ordinary Einstein Lagrangian:

$$L = -(1/\kappa^2) \sqrt{g} R \quad (2)$$

where as usual,  $R$  is the curvature scalar,  $g_{\mu\nu}$  is the metric tensor,  $g$  the determinant of the metric and  $\kappa^2$  is Newton's constant. To this action a Chern-Simons term can be added:

$$L_{CS} = -\frac{i}{4\kappa^2 \mu} \epsilon^{\mu\nu\lambda} (R_{\mu\nu ab} \omega_\lambda^{ab} + \frac{2}{3} \omega_{\mu a}^b \omega_{\nu b}^c \omega_{\lambda c}^a). \quad (3)$$

where

$$R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} + \omega_{\mu a}^c \omega_{\nu cb} - (\mu \leftrightarrow \nu) \quad (4)$$

is the curvature tensor and  $\omega_{\mu ab}$  is the spin connection. The gravitational Chern-Simons charge

$$q_{gr} = \frac{6\pi}{\mu\kappa^2} \quad (5)$$

is quantized and has to be an integer. The presence of matter fields however, fermions and vector bosons with a Chern-Simons term, gives rise to an extra effective contribution to the Chern-Simons charge  $q_{gr}$ .

$$q_{gr}^{ren} = q_{gr} + \frac{1}{8} N_g \text{ sign}(m_g) - \frac{1}{16} N_f \text{ sign}(m_f) \quad (6)$$

where  $N_g$  is the number of vector bosons with topological mass  $m_g$  and  $N_f$  is the number of fermions of mass  $m_f$ . It is important that the corrections are only dependent on the sign of the mass and not its absolute value. This means that also at zero mass an effect is present. Within the purely three dimensional case one speaks therefore of a parity anomaly, since the basic tree level Lagrangian does not violate parity. Embedding the theory in four dimensions with a preferred direction it is easy to understand that the sign is important, since the sign of the mass in the Chern-Simons like term is fixed when one chooses an orientation for the coordinate basis vectors. We now assume that the fundamental gravitational laws have no preferred direction, implying  $q_{gr} = 0$ . The complete effective Chern-Simons term is then induced by the matter fields. In this case the quantization condition gives rise to the following identity

$$N_f \mp 2N_g = 0 \pmod{16} \quad (7)$$

whereby the minus sign is to be taken when the fermions and the bosons have the same sign of the mass. It is assumed that the fermions separately and the bosons separately have the same sign for the mass, which is a reasonable assumption when they are part of the same multiplets in a unified theory, since otherwise one would break the gauge symmetry. We see that the condition (7) is fulfilled for the vector bosons by themselves if the gauge group is  $SU(5)$ , giving  $N_g = 24$  and also for the fermions by themselves, when they are in the 16-dimensional spinor representation of  $SO(10)$ . Moreover it is desirable that the effective renormalized gravitational Chern-Simons charge  $q_{gr}^{ren} = 0$ , since otherwise it is difficult to understand that the late universe is even approximately isotropic, because the gravitational field equations themselves would have a preferred direction. This condition is fulfilled if

there are three generations of fermions  $3 \times 16 - 2 \times 24 = 0$ .

Since ultimately the symmetry of  $SU(5)$  gets broken one can wonder if the consistency condition might play a role in the symmetry breaking pattern. One would expect different signs for the subgroups  $SU(3), SU(2), U(1)$  and the different representations of the fermions under the  $SU(5)$  decomposition  $16 = 10 + \bar{5} + 1$ .

If we take

$$SU(3) \rightarrow +, \quad SU(2) \rightarrow -, \quad U(1) \rightarrow + \quad (8)$$

and

$$10 \rightarrow +, \quad \bar{5} \rightarrow -, \quad 1 \rightarrow - \quad (9)$$

we get  $2 \times (8 - 3 + 1) - 3 \times (10 - 5 - 1) = 0$ . Therefore the chain  $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$  appears possible, however for the chain  $SU(5) \rightarrow SU(4) \times U(1)$  there is no solution, with the above conditions.

We see therefore that the phenomenologically desirable gauge groups and representations are well described by the above conditions. We take this as an indication that the actual topology of space-time is of the form  $M_3 \times U(1)$ . The universe has become very large due to inflation, so the topology is not directly visible in the present epoch. However a remnant of the topology could be the existence of a preferred direction in the structure of the microwave background. An anisotropy at the largest scales may actually have been seen in the WMAP data [23]. A recent analysis of inflation [24, 25] in Bianchi-I models indicates that such an asymmetry could be explained in such a cosmology.

### 3 Discussion

In the above we have presented an argument that determines, or at least constrains, the gauge group and the representations for the fundamental forces in nature. Let us compare our argument with previous attempts in this direction in the literature. An earlier attempt to derive  $SU(5)$  was made in the context of  $N = 8$  supergravity [26]. In order to derive an acceptable model, a number of dynamical assumptions regarding composite states had to be made. With the realization that  $N = 8$  supergravity is not a finite theory and therefore not suitable as a fundamental theory for all forces, attempts along this direction have largely stopped. Another attempt was to use string theory in the form of the heterotic string, which implies a gauge group  $E(8) \times E(8)$  [27]. The reasoning used is somewhat similar to our case. The group is selected by the absence of an anomaly, in this case the conformal anomaly on the world sheet. Subsequently reducing the group to something closer to the standard model appeared possible, but rather complicated. With the realization, that string theory allows for many vacua with different gauge groups, the idea of a unique group has been abandoned in this approach. These two attempts are similar in that they are very ambitious. The assumption is that one determines the unique form of fundamental dynamics from a given mathematical structure, which should subsequently contain the observed forces of nature.

Our attempt is much more modest. We basically combine three apparently unrelated physical inputs in a somewhat surprising way. The inputs are the behaviour of anisotropic universes, the pattern of the quantum numbers of fermions and vector bosons, and the parity

anomaly. All of these have been known for a long time. As an interesting result we found an indication that the universe should be anisotropic at early times. One can wonder how solid the basis of the inputs is. The behaviour of anisotropic, but homogeneous universes has been studied exhaustively and cannot be seriously doubted. The unification pattern of the forces in  $SU(5)$  and the fermions in  $SO(10)$  is quite apparent and convincing. However this pattern can be proven wrong through the discovery of new particles, for instance a  $Z'$  boson, that would enlarge the rank of the gauge group. If this were to happen it is quite difficult to fulfill the constraints that we imposed and would most likely prove the argument to be wrong. The restriction (7), in particular when one requires strict equality, can become quite restrictive, when it is combined with other expectations one might have for a unified field theory. For instance, assuming that the fermions should be in the fundamental representation of a chiral-anomaly free gauge theory, we only found  $E(8)$  with two generations of fermions as a simple example. But here the quantization condition is not satisfied for the generations separately, but only in the presence of both of them. There is somewhat of a controversy[28] regarding the quantization condition in formula (5), since strictly speaking this condition uses a Euclidean formulation for gravity[29]. This is however a general problem in quantum gravity. It is a difficult and unresolved problem within quantum gravity whether a Euclidean formulation is necessary. In ordinary quantum field theory one has of course the Euclidicity postulate, that the Minkowski space Green's functions should be analytic continuations of Euclidean Green's functions, in order to satisfy causality. A last worry is the question whether the gravitational parity anomaly that exists in three dimensions can be lifted to four dimensions as argued in the text. For the Yang-Mills case this was worked out in detail in[15], with the result that there is a direct relation between the three dimensional parity anomaly and the four dimensional CPT anomaly. This derivation can directly be applied to gravity, at least in its linearized form, since in this form there is no fundamental difference between the spin connection and an ordinary gauge field.

Finally we consider the question, whether the inputs can be relaxed and whether extra conditions could still be present. The way the argument was presented, the actual structure of the early universe plays an essential role. This condition might be relaxed within the context of quantum gravity, where topology change is presumably possible. Within a quantum gravitational context it would therefore not be necessary that the actual topology of space-time be  $M_3 \times U(1)$ , but it would be sufficient that this topology is potentially possible. This might lead to possibly stronger conditions, if one imposes, that the spectrum of the matter fields should be such that the theory is anomaly free for all possible compactifications of the actually existing space-time. We wish to point out that our approach to symmetry has some similarity to the one in [30].

## References

- [1] H. Georgi, S.L. Glashow; *Phys. Rev. Lett.* **32**, 438 (1974).
- [2] H. Fritzsch, P. Minkowski; *Ann. Phys. (NY)* **93**, 193 (1975).
- [3] E. Kasner; *Trans. Amer. Math. Soc.* **27**, 155 (1925).

- [4] Dynamical Systems in Cosmology; ed. J. Wainwright, G.F.R. Ellis, Cambridge University Press (1997).
- [5] W. Siegel; *Nucl. Phys. B* **156**, 135 (1979).
- [6] J.F. Schonfeld; *Nucl. Phys. B* **185**, 157 (1981).
- [7] R. Jackiw, S. Templeton; *Phys. Rev. D* **23**, 2291 (1981).
- [8] S. Deser, R. Jackiw, S. Templeton; *Phys. Rev. Lett.* **48**, 975 (1982).
- [9] S. Deser, R. Jackiw, S. Templeton; *Ann. Phys. (NY)* **140**, 372 (1982).
- [10] R.D. Pisarski, S. Rao; *Phys. Rev. D* **32**, 2081 (1985).
- [11] E. Witten; *Phys. Lett. B* **117**, 324 (1982).
- [12] L. Alvarez-Gaumé, E. Witten; *Nucl. Phys. B* **234**, 269 (1983).
- [13] A.N. Redlich; *Phys. Rev. Lett.* **52**, 18 (1984).
- [14] A.N. Redlich; *Phys. Rev. D* **29**, 2366 (1984).
- [15] F.R. Klinkhamer; *Nucl. Phys. B* **578**, 277 (2000).
- [16] F.R. Klinkhamer; in *Proceedings of the 7<sup>th</sup> International Wigner Symposium*, hep-th/**0110135**.
- [17] M.A. Goni, M.A. Valle; *Phys. Rev. D* **34**, 648 (1986).
- [18] I. Vuorio; *Phys. Lett. B* **175**, 176 (1986), Err. B **181**, 416.
- [19] Shuichi Ojima; *Prog. Theor. Phys.* **81**, 512 (1989).
- [20] J.J. van der Bij, R.D. Pisarski, S. Rao; *Phys. Lett. B* **179**, 87 (1986).
- [21] Z. Yang; *Mod. Phys. Lett. A* **5**, 1191 (1990).
- [22] A. Lerda, P. van Nieuwenhuizen; *Phys. Rev. Lett.* **62**, 1217 (1989).
- [23] WMAP Collaboration; *Astrophys. J. Suppl.* **170**, 377 (2007).
- [24] A.E. Gümrükçüoğlu, C.R. Contaldi, M. Peloso; Archiv: **0707.4179**.
- [25] T.S. Pereira, C. Pitrou, J.-P. Uzan; Archiv: **0707.0736**.
- [26] J. Ellis, M.K. Gaillard, B. Zumino; *Phys. Lett. B* **94**, 343 (1980).
- [27] D.J. Gross, J.A. Harvey, E. Martinec, R. Rohm; *Phys. Rev. Lett.* **54**, 502 (1985).
- [28] R. Percacci; *Ann. Phys. (NY)* **177**, 27 (1987).
- [29] Euclidean Quantum Gravity; ed. G.W. Gibbons, S.W. Hawking, World Scientific (1993).
- [30] S. Chadha, H.B. Nielsen; *Nucl. Phys. B* **217**, 125 (1983).